

THE SECOND DIRICHLET COEFFICIENT STARTS OUT NEGATIVE

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Dedicated to the memory of Marvin Knopp.

ABSTRACT. Classical modular forms of small weight and low level are likely to have a negative second Fourier coefficient. Similarly, the labeling scheme for elliptic curves tends to give smaller labels to the higher-rank curves. These observations are easily made when browsing the L-functions and Modular Forms Database, available at <http://www.LMFDB.org/>. An explanation lies in the L-functions associated to these objects.

1. INTRODUCTION

Modular forms and related objects have series expansions which typically are normalized so that the initial coefficient is 1, making the second coefficient the first instance containing useful information. We present data which show that, for several families of objects, the “smallest” members in the family tend to have a negative second coefficient. Here “smallest” refers to a natural ordering of the family. These observations seem surprising at first, because when averaged over the family, the second coefficient has a known (or conjectured) distribution [3, 15, 16] which is equally likely to be positive or negative. The explanation lies in the fact that we can associate an L-function to each object in the family, and we describe a general principle which says that the “first” L-function in a family is likely to have a negative second Dirichlet coefficient. That principle is a straightforward application of Weil’s explicit formula.

There is a substantial literature on bounding the location of the first negative coefficient of modular forms and other objects [2, 8, 10, 11, 13]. The situation we discuss here, in which a negative coefficient appears as soon as possible, should be viewed as a transient phenomenon, and not related to the general question of the oscillation of Fourier coefficients.

In Section 2 we provide data on the first few coefficients of holomorphic cusp forms on Hecke congruence groups, as well as elliptic curves of higher rank. In many cases, the first nontrivial coefficient is negative. In Section 3 we make the argument that the L-functions associated to the objects are responsible for this phenomenon and we use the explicit formula to justify that conclusion.

2. THE DATA

2.1. Classical holomorphic cusp forms. Let $S_k(N)$ denote the space of holomorphic cusp forms of weight k for the Hecke congruence group $\Gamma_0(N) \subset PSL(2, \mathbb{Z})$, and let $S_k^{new}(N)$ denote the subspace of newforms. The newform space has a distinguished basis of simultaneous

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$N =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$k = 2$											•			•	•
4					•	•	•	—	—	◦	2	—	3	2	2
6			•	—	◦	◦	3	1	1	3	4		5	2	4
8		•	◦		3	1	3	2	3	1	6	2	7	4	4
10		◦	2	1	3	1	5	2	3	3	8	1	9	4	6
12	•		◦	1	3	3	5	3	4	5	8	2	11	6	8

TABLE 2.1. The signs of the 2^{nd} Fourier coefficients of newforms in $S_k^{\text{new}}(N)$. The symbol • represents a form with a negative coefficient, ◦ a form with a positive coefficient, — a form with a zero coefficient, and a positive integer is the dimension of the newform space.

eigenfunctions of the Hecke operators and Atkin-Lehner operators. The newforms have a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz},$$

where $a(1) = 1$. A standard reference on this material is Iwaniec’s book [7].

Table 2.1 provides information about the dimension of the space $S_k^{\text{new}}(N)$ and the signs of the Fourier coefficient $a(2)$ of the newform basis. From Table 2.1, one can see that if k and N are both small, then $S_k^{\text{new}}(N)$ is trivial, and the dimension of $S_k^{\text{new}}(N)$ grows regularly as a function of k , and irregularly as a function of N . These facts are well known. But what was previously not observed, and seems surprising at first, is that for most of the newforms which are on the border of the non-zero spaces, $a(2)$ is negative. The negativity of these coefficients is not a coincidence: they arise from a simple principle involving L-functions which we describe in Section 3.1.

While not included in Table 2.1, we note that the second coefficient is negative for $f \in S_2^{\text{new}}(N)$ for all $N \leq 21$. That is, there are four more • along the top row before the first ◦ appears. While this may also seem surprising, it turns out to have a similar explanation, which we give in Section 3.2.

2.2. Elliptic curves. Tables of elliptic curves/ \mathbb{Q} have both inspired and benefitted from significant theoretical work. As of this writing, all elliptic curves of conductor less than 350,000 have been tabulated by John Cremona, and detailed information about them is available in the L-functions and Modular Forms Database (LMFDB) [12].

An issue is how to label elliptic curves. The goal is straightforward: each curve should be given a label which specifies its conductor, its isogeny class, and the isomorphism class within the isogeny class. The conductor is an integer, so there is no ambiguity. The order in which to list the isogeny classes, however, requires a choice, as does the order for listing the curves within an isogeny class.

Various methods for ordering the isogeny classes have been used, and until recently the “Cremona label” has been the standard. Unfortunately, the Cremona label is difficult to describe, and in some cases cannot be derived from first principles. See [4] for details.

class	37.a	43.a	53.a	57.a	58.a	61.a	65.a	77.a	79.a	82.a	88.a	89.a
#N	2	1	1	3	2	1	1	3	1	1	1	2
<hr/>												
91.a and 91.b	92.a	99.a	101.a	102.a	106.a	112.a	117.a	118.a	121.b			
2	2	4	1	3	4	3	1	4	3			

TABLE 2.2. The LMFDB labels of the isogeny classes of the first 23 elliptic curves of rank 1, and the number of isogeny classes $\#N$ with conductor N .

These shortcomings have been addressed by a new labeling scheme, also developed by John Cremona, referred to as the “LMFDB label.” For example, the elliptic curve

$$E : y^2 = x^3 + x^2 + 210x + 1764$$

has LMFDB label “672.e2”. That specific label means:

- E has conductor 672,
- when the newforms with rational integer coefficients in $S_2^{\text{new}}(672)$ are ordered lexicographically by their Fourier coefficients, the form associated to E appears 5th on the list (because “e” is the 5th letter), and
- when the elliptic curves in the isogeny class of E are put in minimal Weierstrass form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

then E appears 2nd when the $[a_1, a_2, a_3, a_4, a_6]$ are ordered lexicographically.

Note that the modularity [1] of elliptic curves/ \mathbb{Q} is necessary to ensure that each curve has an LMFDB label.

One of the interesting properties of an elliptic curve is its *rank*, which is the rank of the Mordell-Weil group $E(\mathbb{Q})$ of rational points on E . The rank of an elliptic curve is the subject of the Birch and Swinnerton-Dyer conjecture, which equates the rank with the order of vanishing of the L-function $L(s, E)$ at the critical point $s = \frac{1}{2}$.

Table 2.2 shows the LMFDB label for the isogeny classes of the first few elliptic curves of rank 1, along with the number of isogeny classes with that conductor. Note that we give the label of an isogeny class, not an elliptic curve, because all the curves in an isogeny class have the same rank.

For the entries in Table 2.2 where there is a single isogeny class of a given conductor, then of course that class has to be given the label “a.” But what seems surprising at first is that even when there are other isogeny classes, the entries in Table 2.2 almost all have the label “a.” In fact, the first 11 times there was a rank 1 curve and also another isogeny class with that same conductor, the rank 1 isogeny class is given the label “a.” That seems highly unlikely, considering that the labeling scheme does not appear to have any reference to the rank.

In Table 2.3 we give the labels and the number of isogeny classes with a given conductor, for the first few elliptic curves of rank 2. We have omitted those cases where there is only one isogeny class with that conductor, such as the first case of rank 2: 389.a.

class	446.a	571.a	664.a	681.a	718.a	794.a	817.a	916.a	994.a
$\#N$	4	2	3	5	3	4	2	5	11

TABLE 2.3. The LMFDB labels of the isogeny classes of the first 9 elliptic curves of rank 2, and the number of isogeny classes $\#N$ with conductor N , for those cases where there is more than one isogeny class for that conductor.

As in the first 11 entries of Table 2.2, every curve in Table 2.3 has isogeny class “a.” In fact, this phenomenon for rank 2 continues until conductor 1147, where isogeny class “b” has rank 2. A similar phenomenon occurs for rank 3, which the interested reader can explore in the LMFDB. We will see that this has a similar explanation to our previous observations.

3. L-FUNCTIONS AND THE EXPLICIT FORMULA

All the observations in Section 2 have the same underlying cause: all are consequences of the following general principle:

L-functions which barely exist tend to have negative coefficients.

In this section we describe what we mean by “L-function” and the sense in which an L-function can “barely exist.” We then show how the above principle explains our observations.

3.1. *L*-functions. By an *L-function*, we mean a Dirichlet series with a functional equation and an Euler product. Furthermore, we assume the Ramanujan-Petersson conjecture and the Generalized Riemann Hypothesis. This means that we can write the *L*-function as a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad (3.1)$$

where $a(n) \ll n^\delta$ for any $\delta > 0$, which has an Euler product

$$L(s) = \prod_p L_p(p^{-s})^{-1}, \quad (3.2)$$

where L_p is a polynomial, and the product is over the primes. As is usual in the theory of L-functions, $s = \sigma + it$ is a complex variable. We assume the Dirichlet coefficients are real, so the L-function satisfies a functional equation which can be written in the form

$$\Lambda(s) = Q^s \prod_{j=1}^d \Gamma\left(\frac{s}{2} + \mu_j\right) L(s) = \varepsilon \Lambda(1-s). \quad (3.3)$$

Here $\varepsilon = \pm 1$ and we assume that $\mu_j \geq 0$ and $Q > 0$. The number d is called the *degree* of the *L*-function, which for all but finitely many p is also the degree of the polynomial L_p . The L-functions associated to the arithmetic objects considered in this paper are conjectured to satisfy the Ramanujan-Petersson bound, which asserts that the polynomials L_p have all

their zeros on or outside the unit circle, so the Dirichlet coefficients actually satisfy the more precise bound

$$|a(p)| \leq d, \quad (3.4)$$

for p prime. Also, the L-functions considered in this paper are conjectured to satisfy the analogue of the Riemann Hypothesis, which says that the zeros of the L-function in the strip $0 < \sigma < 1$ have the form $\rho = \frac{1}{2} + i\gamma$ with $\gamma \in \mathbb{R}$.

A simple but powerful tool in the study of L-functions is Weil's explicit formula.

Lemma 3.1. *Suppose that $L(s)$ has a Dirichlet series expansion (3.1) which continues to an entire function such that*

$$\Lambda(s) = Q^s \prod_{j=1}^d \Gamma\left(\frac{s}{2} + \mu_j\right) L(s) = \varepsilon \Lambda(1-s) \quad (3.5)$$

is entire and satisfies the mild growth condition $L(\sigma + it) \ll |t|^A$ for some $A > 0$, uniformly in t for bounded σ . Let $f(s)$ be an even function which is holomorphic in a horizontal strip $-(1/2 + \delta) < \text{Im}(s) < 1/2 + \delta$ with $f(s) \ll \min(1, |s|^{-(1+\epsilon)})$ in this region, and suppose that $f(x)$ is real-valued for real x . Suppose also that the Fourier transform of f , defined by

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(u) e^{-2\pi i u x} dx,$$

is such that

$$\sum_{n=1}^{\infty} \frac{c(n)}{n^{1/2}} \hat{f}\left(\frac{\log n}{2\pi}\right)$$

converges absolutely, where $c(n)$ is defined by

$$\frac{L'}{L}(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}. \quad (3.6)$$

Then

$$\sum_{\gamma} f(\gamma) = \frac{\hat{f}(0)}{\pi} \log Q + \frac{1}{2\pi} \sum_{j=1}^d \ell(\mu_j, f) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{c(n)}{n^{1/2}} \hat{f}\left(\frac{\log n}{2\pi}\right), \quad (3.7)$$

where

$$\ell(\mu, f) = \text{Re} \left\{ \int_{\mathbb{R}} \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \left(\frac{1}{2} + it \right) + \mu \right) f(t) dt \right\} - \hat{f}(0) \log \pi, \quad (3.8)$$

and the sum \sum_{γ} runs over all non-trivial zeroes of $L(s)$.

Proof. This can be found in Iwaniec and Kowalski [9], page 109, but note that they use a different normalization for the Fourier transform. \square

The following is a precise version of the principle stated at the beginning of this section.

Theorem 3.2. *Fix non-negative real numbers μ_1, \dots, μ_d in (3.3). There exist real numbers $0 < Q_0 < Q_1$ such that if $0 < Q < Q_0$ then there do not exist any L-functions with functional equation (3.3) satisfying the Ramanujan bound (3.4) and the Riemann Hypothesis. And if $Q_0 < Q < Q_1$ then any L-function satisfying those three conditions must have $a(2) < 0$.*

In other words, if an L-function is just barely able to exist, meaning that the parameter Q in its functional equation is only slightly larger than the minimum threshold imposed by the μ_j , then it must have a negative second Dirichlet coefficient.

The proof involves choosing an appropriate test function in the explicit formula. We will make a simple choice in order to illustrate the method, making no attempt to obtain optimal results.

Proof of Theorem 3.2. In the explicit formula, choose the function

$$f(x) = \frac{1}{2\pi} \frac{\sin^2(x/2)}{(x/2)^2}, \quad (3.9)$$

which satisfies

$$\hat{f}(x) = \begin{cases} 1 - 2\pi|x| & \text{if } -\frac{1}{2\pi} < x < \frac{1}{2\pi} \\ 0 & \text{otherwise.} \end{cases} \quad (3.10)$$

The key properties we require are that f is non-negative, and the support of \hat{f} contains $\log(2)/(2\pi)$ but not $\log(n)/(2\pi)$ for any integer $n > 2$, and $\hat{f}(\log(2)/(2\pi)) > 0$.

Substituting into (3.7) we obtain

$$\sum_{\gamma} f(\gamma) = \frac{1}{\pi} \log Q + \frac{1}{2\pi} \sum_{j=1}^d \ell(\mu_j, f) + 0.069066 c(2). \quad (3.11)$$

The sum on the left side is non-negative, and the sum over $\ell(\mu_j, f)$ is a constant depending on the μ_j and our choice of f . The Ramanujan bound on the Dirichlet coefficients gives us a bound on $c(2)$, coming from the simple fact that for p prime, $c(p) = -a(p) \log(p)$ where $a(p)$ is the p th Dirichlet coefficient of the L-function.

Thus, if Q is sufficiently small then the right side must be negative, which is a contradiction. That proves the existence of the number Q_0 . And if Q is slightly larger, then the only way for the right side to be non-negative is if $0.069066 c(2)$ is positive; this is equivalent to $a(2)$ being negative. That proves the existence of the number Q_1 . \square

To complete our explanation of the prevalence of $a(2) < 0$ for cusp forms on the edge of existence, we note that for the L-functions associated to $F \in S_k^{new}(N)$, the parameters Q and μ in the functional equation are given by $Q = N/\pi$ and $\{\mu_1, \mu_2\} = \{\frac{k-1}{4}, \frac{k+1}{4}\}$. Applying Theorem 3.2 with k fixed, we see that if N is sufficiently small then $S_k^{new}(N)$ must be empty, and if N is slightly larger and $S_k^{new}(N)$ is nonempty, then $a(2) < 0$.

For the elliptic curves of higher rank, we note that, according to the Birch and Swinnerton-Dyer conjecture, the L-function of a rank r elliptic curve has an order r zero at the critical point. This translates to r zeros with $\gamma = 0$ in (3.7). Those $\gamma = 0$ terms add a large positive contribution to the left side of (3.7), equivalently (3.11). Thus, the lower bound on possible levels $N = \pi Q$ for a rank r elliptic curve is an increasing function of r . And just as in the proof of Theorem 3.2, those Q which are slightly larger than the minimum must be accompanied by a negative value for $a(2)$. Finally, since the isogeny classes of elliptic curves are ordered lexicographically by the L-function (equivalently, modular form) coefficients,

those with a negative $a(2)$ are more likely to be listed first, receiving “a” as the label of their isogeny class.

3.2. Small weight. For the larger weights in Table 2.1, the $a(2) < 0$ phenomenon only persists for a small strip around the edge of the region where the cusp forms are able to exist. However, for weight $k = 2$ we find that $a(2) < 0$ for all $N \leq 21$. This also has a simple explanation coming from the explicit formula.

For a weight k cusp form, the parameters $\{\mu_1, \mu_2\}$ in the functional equation are $\{\frac{k-1}{4}, \frac{k+1}{4}\}$. In Figure 3.1 we plot the factor

$$\operatorname{Re} \left[\frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \left(\frac{1}{2} + it \right) + \mu \right) \right] \quad (3.12)$$

which occurs in the $\ell(\mu, f)$ term in the explicit formula, (3.8), for various μ .

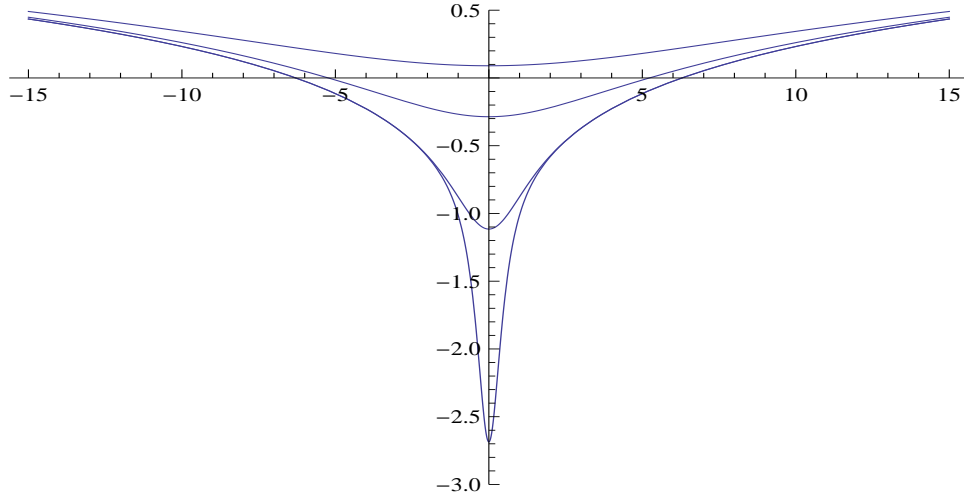


FIGURE 3.1. Plots of the real part of $\frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \left(\frac{1}{2} + it \right) + \mu \right)$ for $\mu = 0, 1, 4$, and 8 , where smaller values of μ correspond to lower graphs.

In the integrand of (3.8), the function in Figure 3.1 is multiplied by $f(t)$, which is positive and has most of its support near $t = 0$. Therefore the integrand is mostly negative when the weight k is small, so the contribution of $\ell(\mu, f)$ to the right side of the explicit formula will be negative. Thus, for small k the value of $Q = \pi N$ must be larger in order for the right side of the explicit formula to be non-negative. And as before, for Q slightly above the threshold value it is necessary for $a(2)$ to be negative. Let $-\delta$ be the negative amount which $a(2)$ could contribute to the right side. Since it is $\log Q$ that contributes to the right side, one needs $\log Q$ to increase by δ before it is possible for $a(2)$ to be positive. But if Q is larger, as it must be if k is small, then $\log Q$ increases more slowly, so one requires a larger increase in Q to increase $\log Q$ by δ . This suggests that if k is small, once Q is large enough for the curve (or the L-function) to possibly exist, there is a wider range of Q values which require $a(2)$ to be negative. That explains the large number of \bullet at the top of Table 2.1.

It is a curious phenomenon that elliptic curves of a given rank tend to appear with conductors that are only slightly larger (on a logarithmic scale) than the minimum forced upon

them by the explicit formula. This means that many of the initial coefficients, and not just $a(2)$, must be negative. This fact imparts some surprising properties on the L-functions. See Michael Rubinstein's paper [14] and its references for further discussion, data, and plots.

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